INDICATORS FOR PLURISUBHARMONIC FUNCTIONS OF LOGARITHMIC GROWTH

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Abstract. A notion of indicator for a plurisubharmonic function u of logarithmic growth in $\mathbf{C}^n$ is introduced and studied. It is applied to evaluation of the total Monge-Ampère measure $(dd^c u)^n(\mathbf{C}^n)$. Upper bounds for the measure are obtained in terms of growth characteristics of u. When $u = \log|f|$ for a polynomial mapping f with isolated zeros, the indicator generates the Newton polyhedron of f whose volume bounds the number of the zeros.

1 Introduction

We consider plurisubharmonic functions u of logarithmic growth in $\mathbf{C}^n$, i.e. satisfying the relation

$$u(z) \le C_1 \log^+ |z| + C_2 \tag{1}$$

with some constants $C_j = C_j(u) \ge 0$. The class of such functions will be denoted by $\mathcal{L}(\mathbf{C}^n)$ or simply by $\mathcal{L}$. (It is worth mentioning that in the literature the notation $\mathcal{L}$ is used sometimes for the class of functions satisfying (1) with $C_1 = 1$; for our purposes we need to consider the whole class of functions of logarithmic growth, and denoting it by $\mathcal{L}$ we follow, for example, [13], [14].) It is an important class containing, in particular, functions of the form $\log|P|$ with polynomial mappings $P : \mathbf{C}^n \to \mathbf{C}^N$. Various results concerning the functions of logarithmic growth can be found in [10]-[14], [20], [6], see also the references in [6] and [5]. For general properties of plurisubharmonic functions and the complex Monge-Ampère operators, we refer the reader to [11], [16], [6], and [3].

A remarkable property of functions $u \in \mathcal{L}$ is finiteness of their total Monge-Ampère measures

$$M(u; \mathbf{C}^n) = \int_{\mathbf{C}^n} (dd^c u)^n < \infty$$

as long as $(dd^c u)^n$ is well defined on the whole $\mathbf{C}^n$; we use the notation $d = \partial + \bar\partial$, $d^c = (\partial - \bar\partial)/2\pi i$. Moreover, the total mass is tied strongly to the growth of the function. For example, if

$$\log^+ |z| + c \le u(z) \le \log^+ |z| + C, \tag{2}$$

then $M(u; \mathbf{C}^n) = M(\log|z|; \mathbf{C}^n) = 1$. The objectives for the present paper is to study $M(u; \mathbf{C}^n)$ when no regularity condition on u like (2) is assumed. In case of

$u = \log|P|$ with $P : \mathbf{C}^n \to \mathbf{C}^N$ a polynomial mapping with isolated zeros, $M(u;\mathbf{C}^n)$ equals (if $N = n$) or dominates (if $N > n$) the number of the zeros counted with their multiplicities.

If $u = v$ near the boundary of a bounded pseudoconvex domain Ω, then

$$\int_\Omega (dd^c u)^n = \int_\Omega (dd^c v)^n,$$

so the total measure $M(u;\mathbf{C}^n)$ is determined by the asymptotic behavior of u at infinity. For its evaluation we thus need precise characteristics of the behavior. The basic one is the *logarithmic type*

$$\sigma(u) = \limsup_{z\to\infty} \frac{u(z)}{\log|z|}. \tag{3}$$

Another known characteristic is the *logarithmic multitype* $(\sigma_1(u),\ldots,\sigma_n(u))$ [14]:

$$\sigma_1(u) = \sup\{\tilde\sigma_1(u;z') : z' \in \mathbf{C}^{n-1}\} \tag{4}$$

where $\tilde\sigma_1(u;z')$ is the logarithmic type of the function $u_{1,z'}(z_1) = u(z_1,z') \in \mathcal{L}(\mathbf{C})$ with $z' \in \mathbf{C}^{n-1}$ fixed, and similarly for $\sigma_2(u),\ldots,\sigma_n(u)$. For example, if P is a polynomial of degree d_k in z_k, then $\sigma_k(\log|P|) = d_k$.

Due to the certain symmetry between the behavior of $u \in \mathcal{L}$ at infinity and the local behavior of a plurisubharmonic function at a fixed point of its logarithmic singularity, the type $\sigma(u)$ can be regarded as the Lelong number of u at infinity:

$$\sigma(u) = \nu(u,\infty). \tag{5}$$

One can also consider the directional Lelong numbers at infinity with respect to directions $a = (a_1,\ldots,a_n) \in \mathbf{R}^n_+$:

$$\nu(u,a,\infty) = \limsup_{z\to\infty} \frac{u(z)}{S_a(z)}, \tag{6}$$

where

$$S_a(z) = \sup_k a_k^{-1}\log|z_k|. \tag{7}$$

In [17], the residual Monge-Ampère measure of a plurisubharmonic function u at a point $x \in \mathbf{C}^n$, $(dd^c u)^n|_{\{x\}}$, was studied by means of the local indicator of u at x. Using the same approach, we introduce here a notion of the indicator of $u \in \mathcal{L}$:

$$\Psi_{u,x}(y) = \lim_{R\to+\infty} R^{-1}\sup\{u(z) :\ |z_k - x_k| \le |y_k|^R,\ 1 \le k \le n\}.$$

It is a plurisubharmonic function of the class $\mathcal{L}$ which is the (unique) logarithmic tangent to u at x, i.e. the weak limit in $L^1_{loc}(\mathbf{C}^n)$ of the functions $m^{-1}u(x_1 +$

$y_1^m, \dots, x_n + y_n^m)$ as $m \to \infty$ (Theorem 2). The above characteristics of u can be easily expressed in terms of its indicator (see Proposition 3); moreover, it controls the behavior of u in the whole $\mathbf{C}^n$ (Theorem 1):

$$u(z) \le \Psi_{u,x}(x - x) + C_x \quad \forall z \in \mathbf{C}^n.$$

If $(dd^c u)^n$ is defined on $\mathbf{C}^n$, the indicator also controls the total Monge-Ampère mass of u (Theorem 4):

$$M(u; \mathbf{C}^n) \le M(\Psi_{u,x}; \mathbf{C}^n). \tag{8}$$

Since $\Psi_{u,x}(y) = \Psi_{u,x}(|y_1|, \dots, |y_n|)$, the evaluation of its mass is much more easy than that for the original function u. It gives us, in particular, the bounds

$$M(u; \mathbf{C}^n) \le \frac{[\nu(u, a, \infty)]^n}{a_1 \dots a_n} \quad \forall a \in \mathbf{R}^n_+$$

(Theorem 5) and

$$M(u; \mathbf{C}^n) \le n!\, \sigma_1(u) \dots \sigma_n(u)$$

(Theorem 8). A particular case of the latter result (when u is the logarithm of modulus of an equidimensional polynomial mapping with isolated zeros of regular multiplicities) was obtained in [15].

In Theorems 6 and 7 we give a geometric description for the mass of an indicator. Denote $\psi_{u,x}(t) = \Psi_{u,x}(e^{t_1}, \dots, e^{t_n})$, $t = (t_1, \dots, t_n) \in \mathbf{R}^n$, and

$$\Theta_{u,x} = \{a \in \mathbf{R}^n : \langle a, t \rangle \le {\psi_{u,x}}^+(t) \ \forall t \in \mathbf{R}^n\}.$$

Then

$$M(\Psi_{u,x}; \mathbf{C}^n) = n!\, Vol(\Theta_{u,x}). \tag{9}$$

When $u = \log |P|$ with P a polynomial mapping, the set $\Theta_{u,0}$ is the Newton polyhedron for P at infinity (see, for example, [8]), i.e. the convex hull of the set $\omega_0 \cup \{0\}$,

$$\omega_0 = \{s \in \mathbf{Z}^n_+ : \sum_j \left| \frac{\partial^s P_j}{\partial z^s}(0) \right| \ne 0\},$$

and so the right-hand side of (9) is the Newton number of P at infinity. Therefore, a hard result due to A.G. Kouchnirenko on the number of zeros of an equidimensional polynomial mapping [7] follows directly from (8) and (9).

2 Indicators as growth characteristics

Let u be a plurisubharmonic function in $\mathbf{C}^n$. Given $x \in \mathbf{C}^n$ and $t \in \mathbf{R}^n$, denote by $g(u, x, t)$ the mean value of u over the set $T_t(x) = \{z \in \mathbf{C}^n : \ |z_k - x_k| = e^{t_k}, \ 1 \le k \le n\}$, and by $g'(u, x, t)$ the maximum of u on $T_t(x)$.

Proposition 1 *Let $u \in \mathcal{L}$, $x \in \mathbf{C}^n$. Then for every $t \in \mathbf{R}^n$ the following limits exist and coincide:*

$$\lim_{R\to+\infty} R^{-1}g(u,x,Rt) = \lim_{R\to+\infty} R^{-1}g'(u,x,Rt) =: \psi_{u,x}(t) < \infty.$$

Moreover, if $g'(u,x,0) \le 0$, the common limit $\psi_{u,x}(t)$ is obtained by the increasing values.

Proof. For $x \in \mathbf{C}^n$ and $t \in \mathbf{R}^n$ fixed, the function $f(R) := g(u,x,Rt)$ is convex on $\mathbf{R}$ and has the bound $f(R) \le C_1R + C_2$ $\forall R > 0$ with some $C_1, C_2 > 0$. Therefore, for all $R_0 \in \mathbf{R}$, the ratio

$$\frac{f(R)-f(R_0)}{R-R_0} \tag{10}$$

is increasing in $R > R_0$ and bounded and thus has a limit as $R \to +\infty$. It implies the existence of $\hat{g}(u,x,t) = \lim_{R\to+\infty} R^{-1}g(u,x,Rt)$. In the same way we get the value $\hat{g}'(u,x,t) = \lim_{R\to+\infty} R^{-1}g'(u,x,Rt)$. Evidently, $\hat{g}(u,x,t) \le \hat{g}'(u,x,t)$, and the standard arguments using Harnack's inequality give us $\hat{g}(u,x,t) = \hat{g}'(u,x,t)$. The last statement of the proposition follows from the increasing of (10) with $R_0 = 0$.

Now we proceed, as in [17], to a plurisubharmonic characteristic of growth for $u \in \mathcal{L}$. Denote $\mathbf{C}^{*n} = \{z \in \mathbf{C}^n : z_1 \dots z_n \ne 0\}$. The mappings $Log : \mathbf{C}^{*n} \to \mathbf{R}^n$ and $Exp : \mathbf{R}^n \to \mathbf{C}^{*n}$ are defined as $Log(z) = (\log|z_1|, \dots, \log|z_n|)$ and $Exp(t) = (\exp t_1, \dots, \exp t_n)$, respectively. Let $\mathcal{L}^c$ be the subclass of $\mathcal{L}$ formed by n-circled plurisubharmonic functions u, i.e. $u(z) = u(|z_1|, \dots, |z_n|)$. By $\mathcal{L}(\mathbf{R}^n)$ we denote the class of functions $\varphi(t)$, $t \in \mathbf{R}^n$, which are convex in t, increasing in each t_k and such that there exists a limit $\lim_{T\to+\infty} T^{-1}\varphi(T,\dots,T) < \infty$.

The mappings Exp and Log generate an isomorphism between the cones $\mathcal{L}^c$ and $\mathcal{L}(\mathbf{R}^n)$ ([14], Th. 1): $u \in \mathcal{L}^c \iff Exp^*u \in \mathcal{L}(\mathbf{R}^n)$, $h \in \mathcal{L}(\mathbf{R}^n) \iff Log^*h$ extends to a (unique) function from the class $\mathcal{L}^c$. Given $u \in \mathcal{L}^c$, the function Exp^*u will be referred to as the *convex image* of u.

If $h = Exp^*u \in \mathcal{L}(\mathbf{R}^n)$ satisfies the homogeneity condition

$$h(ct) = ch(t) \quad \forall c > 0,\ \forall t \in \mathbf{R}^n,$$

the function u will be called *an indicator*. We denote the collection of all indicators by $\mathcal{I}$. It is easy to see that any indicator Ψ satisfies $\Psi \le 0$ in the unit polydisk

$$D = \{z \in \mathbf{C}^n :\ |z_k| < 1,\ 1 \le k \le n\}$$

and $\Psi > 0$ in

$$D^{-1} = \{z \in \mathbf{C}^n :\ |z_k| > 1,\ 1 \le k \le n\}$$

if $\Psi \not\equiv 0$.

Clearly, the function $\psi_{u,x}$ defined in Proposition 1 belongs to the class $\mathcal{L}(\mathbf{R}^n)$, so $Log^*\psi_{u,x}$ extends to a function $\Psi_{u,x} \in \mathcal{L}^c$:

$$\Psi_{u,x}(y) = \psi_{u,x}(\log|y_1|, \ldots, \log|y_n|), \ y \in \mathbf{C}^{*n}.$$

Moreover, $\Psi_{u,x} \in \mathcal{I}$. We will call it *the indicator of* $u \in \mathcal{L}$ *at* x.

The restriction of $\Psi_{u,x}$ to the polydisk D coincides with the local indicator of u at x introduced in [17]. In particular, $\Psi_{u,x} \equiv 0$ in D if and only if the Lelong number of u at x equals 0. Besides, the directional Lelong numbers of $\Psi_{u,x}$ at 0 are the same as those of u at x.

Proposition 2 *Let* $\Phi \in \mathcal{I}$, *then*

(a) Φ *is continuous as a function* $\mathbf{C}^n \to \mathbf{R} \cup \{-\infty\}$;

(b) $\Psi_{\Phi,x}(y) = \Phi(\tilde{y})$ *where* $\tilde{y}_k = \sup\{|y_k|, 1\}$ *if* $x_k \neq 0$, *and* $\tilde{y}_k = y_k$ *otherwise.*

Proof. *(a)* Since $Exp^*\Phi \in C(\mathbf{R}^n)$, $\Phi \in C(\mathbf{C}^{*n})$. Its continuity on $\mathbf{C}^n$ can be shown by induction in n. Let it be already proved for $n \le l$ (the case $n = 1$ is obvious). Consider any point $z^0 \in \mathbf{C}^{l+1}$ with $z_j^0 = 0$ for some j. If $\Phi(z^0) = -\infty$, then $\Phi(z^s) \to -\infty$ for every sequence $z^s \to z^0$. If $\Phi(z^0) > -\infty$, consider the projections $\tilde{z}^s$ of $z^s \to z^0$ to the subspace $L_j = \{z \in \mathbf{C}^{l+1} : z_j = 0\}$: $\tilde{z}_j^s = 0$ and $\tilde{z}_m^s = z_m^s \ \forall m \neq j$. Since $\Phi|_{L_j} \not\equiv -\infty$, the induction assumption implies $\Phi(\tilde{z}^s) \to \Phi(z^0)$. Therefore, $\liminf \Phi(z^s) \ge \liminf \Phi(\tilde{z}^s) = \Phi(z^0)$ that proves lower semicontinuity of Φ at z^0 and thus its continuity.

(b) For any $t \in \mathbf{R}^n$ and $R > 0$,

$$\begin{aligned} R^{-1}g'(\Phi, x, Rt) &= R^{-1}\Phi(|x_1| + e^{Rt_1}, \ldots, |x_n| + e^{Rt_n}) \\ &= \Phi([|x_1| + e^{Rt_1}]^{1/R}, \ldots, [|x_n| + e^{Rt_n}]^{1/R}). \end{aligned}$$

The argument $[|x_k| + e^{Rt_k}]^{1/R}$ tends to $\exp\{t_k^+\}$ if $x_k \neq 0$, and to $\exp\{t_k\}$ otherwise, so the statement follows from *(a)*.

The growth characteristics (3), (4), (6) of functions $u \in \mathcal{L}$ can be expressed in terms of the indicators. We will use the following notation:

$$\mathbf{1} = (1, \ldots, 1), \ \mathbf{1}_1 = (1, 0, \ldots, 0), \ \mathbf{1}_2 = (0, 1, 0, \ldots, 0), \ \ldots, \mathbf{1}_n = (0, \ldots, 0, 1). \quad (11)$$

Proposition 3 *(a)* $\nu(u, \infty) = \nu(\Psi_{u,x}, \infty) = \psi_{u,x}(\mathbf{1})$;

(b) $\nu(u, a, \infty) = \nu(\Psi_{u,x}, a, \infty) = \psi_{u,x}(a) \quad \forall a \in \mathbf{R}_+^n$;

(c) $\sigma_k(u) = \sigma_k(\Psi_{u,x}) = \psi_{u,x}(\mathbf{1}_k), \ k = 1, \ldots, n.$

Proof. The relation $\nu(u,a,\infty)=\psi_{u,x}(a)$ follows directly from the definition of $\psi_{u,x}$. The equalities $\nu(u,\infty)=\psi_{u,x}(\mathbf{1})$ and $\sigma_k(u)=\psi_{u,x}(\mathbf{1}_k)$ are proved in Theorems 1 and 2 of [14]. Being applied to the function $\Psi_{u,x}$ instead of u, they give us the first equalities in *(a)–(c)* in view of Proposition 2. The proof is complete.

Theorem 1 *Let $u\in\mathcal{L}$, $x\in\mathbf{C}^n$. Then*

$$u(z)\le \Psi_{u,x}(z-x)+C\quad \forall z\in\mathbf{C}^n \tag{12}$$

with $C=g'(u,x,0)$. Moreover, $\Psi_{u,x}$ is the least indicator satisfying (12) with some constant C.

Proof. By Proposition 1,

$$g'(u,x,Rt)\le R\,\psi_{u,x}(t)+g'(u,x,0)\quad \forall R>0,\ \forall t\in\mathbf{R}^n,$$

that implies (12) since $u(z+x)\le g'(u,x,Log(z))$.

If $\Phi\in\mathcal{I}$ satisfies $u(z)\le\Phi(z-x)+C$, then

$$\Psi_{u,x}\le\Psi_{\Phi(\cdot+x),x}=\Psi_{\Phi,0}=\Phi,$$

the latter equality being a consequence of Proposition 2. The theorem is proved.

The indicator $\Psi_{u,x}$ can be easily calculated in the algebraic case, i.e. when u is the logarihm of modulus of a polynomial mapping. Recall that *the index* $I(P,x,a)$ of a polynomial P at $x\in\mathbf{C}^n$ with respect to the weight $a\in\mathbf{R}^n_+$ is defined as

$$I(P,x,a)=\inf\{\langle a,J\rangle:\ J\in\omega_x\}$$

where

$$\omega_x=\{J\in\mathbf{Z}^n_+:\ \frac{\partial^J P}{\partial z^J}(x)\ne 0\}$$

(see e.g. [9]). For any $t\in\mathbf{R}^n$ we define

$$I_{up}(P,x,t)=\sup\{\langle t,J\rangle:\ J\in\omega_x\}, \tag{13}$$

the *upper index* of P at $x\in\mathbf{C}^n$ with respect to $t\in\mathbf{R}^n$. Clearly, $I_{up}(P,x,t)=-I(P,x,-t)$ for all $t\in-\mathbf{R}^n_+$.

Proposition 4 *Let $u=\log|P|$, $P:\mathbf{C}^n\to\mathbf{C}$ being a polynomial. Then*

$$\psi_{u,x}(t)=I_{up}(P,x,t)\quad \forall t\in\mathbf{R}^n,\ \forall x\in\mathbf{C}^n.$$

Proof. Let

$$P(z) = \sum_{J\in\omega_x} c_J\,(z-x)^J$$

and $d = I_{up}(P,x,t)$, so $b_J := \langle t, J\rangle - d \le 0\ \forall J \in \omega_x$. Then

$$R^{-1}g'(u,x,t) = d + R^{-1}\sup_\theta \{\log|\sum_J c_J \exp[Rb_J i\langle\theta, J\rangle]|\}.$$

Since there exists $J_0 \in \omega_x$ with $b_{J_0} = 0$, the second term here tends to 0 as $R \to +\infty$, and the statement follows.

Proposition 5 *Let $u_1,\ldots,u_m \in \mathcal{L}$, $u = \sup_k u_k$, $v = \log\sum_k \exp u_k$. Then*

$$\Psi_{u,x} = \Psi_{v,x} = \sup_k\ \Psi_{u_k,x}.$$

Proof. Since $u \ge u_k$, we have $\Psi_{u,x} \ge \sup_k\ \Psi_{u_k,x}$. On the other hand, by (12),

$$u(z) \le \sup_k \{\Psi_{u_k,x} + C_k\} \le \sup_k\ \Psi_{u_k,x} + \sup_k\ C_k,$$

and the equality $\Psi_{u,x} = \sup_k\ \Psi_{u_k,x}$ results from Theorem 1.

Similarly, the relations $\Psi_{v,x} \ge \Psi_{u,x}$ and

$$v(z) \le \log\sum_k \exp[\Psi_{u_k,x}(z-x) + C_k] \le \Psi_{u,x}(z-x) + m + \sup_k\ C_k$$

imply $\Psi_{u,x} = \Psi_{v,x}$, and the proof is complete.

As a corollary of Propositions 4 and 5 we get

Proposition 6 *Let*

$$u = \frac{1}{q}\log\sum_{k=1}^m |P_k|^q$$

with $P_1,\ldots,P_m$ polynomials and $q > 0$. Then $\psi_{u,x}(t) = \sup_k\ I_{up}(P_k,x,t)$.

The indicator $\Psi_{u,x}$ can be described as a tangent (in logarithmic coordinates) to the original function $u \in \mathcal{L}$. For $z \in \mathbf{C}^n$ and $m \in \mathbf{N}$, we set $z^m = (z_1^m,\ldots,z_n^m)$ and define the function

$$(\mathcal{T}_{m,x}u)(z) = m^{-1}u(x+z^m) \in \mathcal{L}.$$

Theorem 2 *$\mathcal{T}_{m,x}u \to \Psi_{u,x}$ in $L^1_{loc}(\mathbf{C}^n)$ as $m \to +\infty$.*

Proof. First, the family $\{\mathcal{T}_{m,x}u\}_m$ is relatively compact in $L^1_{loc}(\mathbf{C}^n)$. Really, (12) implies

$$(\mathcal{T}_{m,x}u)(z) \leq \Psi_{u,x}(z-x) + m^{-1}C \quad \forall m. \tag{14}$$

Therefore, the family is uniformly bounded above on each compact subset of $\mathbf{C}^n$. Besides, $g(\mathcal{T}_{m,x}u,0,0) = m^{-1}g(u,x,0) \to 0$ and hence $g(\mathcal{T}_{m,x}u,0,0) \geq -1$ for all $m \geq m_0$, and the compactness follows.

Now let v be a partial weak limit of $\mathcal{T}_{m,x}u$, i.e. $\mathcal{T}_{m_s,x}u \to v$ for some subsequence m_s. By (14),

$$v \leq \Psi_{u,x}. \tag{15}$$

On the other hand, the convergence of $\mathcal{T}_{m_s,x}u$ to v implies

$$g(\mathcal{T}_{m_s,x}u,0,t) \to g(v,0,t) \quad \forall t \in \mathbf{R}^n.$$

At the same time, by the definition of $\psi_{u,x}$,

$$g(\mathcal{T}_{m_s,x}u,0,t) = m^{-1}g(u,x,mt) \to \psi_{u,x}(t) \quad \forall t \in \mathbf{R}^n,$$

so $g(v,0,t) = \psi_{u,x}(t)$ and thus $g(v,0,t) = g(\Psi_{u,x},0,t)$. Being compared to (15) it gives us $v = \Psi_{u,x}$, that completes the proof.

We conclude this section by studying dependence of $\Psi_{u,x}$ on x.

Proposition 7 *Let $u \in \mathcal{L}$. Then*

(a) $\Psi_u(z) := \sup\{\Psi_{u,x}(z) : x \in \mathbf{C}^n\} \in \mathcal{L}$;

(b) for any $z \in \mathbf{C}^n$, $\Psi_{u,x}(z) = \Psi_u(z)$ for all $x \in \mathbf{C}^n \setminus E_z$, E_z being a pluripolar subset of $\mathbf{C}^n$;

(c) for any $z \in D^{-1}$, $\Psi_{u,x}(z) = \Psi_u(z)$ for all $x \in \mathbf{C}^n$;

(d) $\Psi_u(z) \geq 0 \quad \forall z \in \mathbf{C}^n, \quad \Psi_u \equiv 0$ *in D.*

Proof. Since $u \in \mathcal{L}$, there is a constant $A > 0$ such that $u(z) \leq A\,S^+(z)\ \forall z \in \mathbf{C}^n$, where $S^+(z) = S_1^+(z) = \sup_k \log^+|z_k|$.

We fix a point $z \in \mathbf{C}^n$ and consider the function

$$u_R(x) = R^{-1}g'(u,x,R\,Log(z)), \quad R > 0.$$

It is plurisubharmonic in $\mathbf{C}^n$, and

$$u_R(x) \leq R^{-1}A\,S^+(|x_1| + |z_1|^R, \ldots, |x_n| + |z_n|^R).$$

Therefore, the family $\{u_R\}_{R>1}$ is uniformly bounded above on compact subsets of $\mathbf{C}^n$, and

$$u_\infty(x) := \limsup_{R\to\ +\infty} u_R(x) \le A\, S^+(z) \quad \forall x \in \mathbf{C}^n. \tag{16}$$

Its regularization $u_\infty^*(x) = \limsup_{y\to x} u_\infty(y)$ is then plurisubharmonic in $\mathbf{C}^n$ and bounded and so $u_\infty^* = const.$ We have $u_\infty(x) \le u_\infty^*(x)$ for all $x \in \mathbf{C}^n$ with the equality outside a pluripolar set $E_z \subset \mathbf{C}^n$. We observe now that $u_\infty(x) = \Psi_{u,x}(z)$ and $u_\infty^*(x) = \Psi_u(z)$, so *(b)* is proved.

Let $z^{(j)} \to z$, then the set

$$E = \bigcup_{j=1}^{\infty} E_{z^{(j)}} \cup E_z$$

is pluripolar. For $x \in \mathbf{C}^n \setminus E$,

$$\Psi_u(z) = \Psi_{u,x}(z) = \lim_{j\to\infty} \Psi_{u,x}(z^{(j)}) = \lim_{j\to\infty} \Psi_u(z^{(j)}),$$

that proves continuity of Ψ_u. Therefore, $\Psi_u = \Psi_u^*$ is plurisubharmonic and belongs to $\mathcal{L}$ in view of (16), that gives us *(a)*.

If $x,\ y \in \mathbf{C}^n$ and $a \in \mathbf{R}_+^n$, we have for any $\epsilon > 0$, $g'(u,x,Ra) \le g'(u,y,(1+\epsilon)Ra)$ for all $R > R_0(\epsilon,x,y)$, so $\psi_{u,x}(a) \le \psi_{u,y}(a)$ that implies *(c)*.

Finally, *(d)* follows from the relation $\Psi_{u,x}|_D = 0$ provided $\nu(u,x) = 0$.

3 Monge-Ampère measures

Now we pass to study the Monge-Ampère measures of functions $u \in \mathcal{L}$. We can benefit by the plurisubharmonicity of the growth characteristic Ψ_u as well as by its specific properties established in the previous section.

Any indicator Φ belongs to $L^\infty_{loc}(\mathbf{C}^{*n})$, so $(dd^c\Phi)^n$ is well defined on $\mathbf{C}^{*n}$. If $\Phi \in L^\infty_{loc}(\mathbf{C}^n \setminus \{0\})$, then $(dd^c\Phi)^n$ is defined on the whole space $\mathbf{C}^n$; the class of such indicators will be denoted by $\mathcal{I}_0$.

Let T denote the distinguished boundary $\{z \in \mathbf{C}^n : |z_1| = \ldots = |z_n| = 1\}$ of the unit polydisk D.

Proposition 8 *Let $\Phi \in \mathcal{I}$. Then*

(a) $(dd^c\Phi)^n = 0$ *on* $\mathbf{C}^{*n} \setminus T$*;*

(b) if $\Phi \in \mathcal{I}_0$*, then* $(dd^c\Phi)^n = \tau'_\Phi\, \delta(0) + \tau''_\Phi\, dm_T$ *where* $\tau'_\Phi, \tau''_\Phi \ge 0$*,* $\delta(0)$ *is the Dirac measure at* 0*, and* $dm_T = (2\pi)^{-n} d\theta_1 \ldots d\theta_n$ *is the normalized Lebesgue measure on* T*.*

Proof. *(a)* It suffices to show that for every $y \in \mathbf{C}^{*n}$ there exists an analytic disk γ_y containing y such that the restriction of Φ to γ_y is harmonic near y ([4], Lemma 6.9). Let $y = (|y_1|e^{i\theta_1}, \ldots, |y_n|e^{i\theta_n}) \in \mathbf{C}^{*n}$. Consider the mapping $\lambda_y : \mathbf{C} \to \mathbf{C}^n$ given by

$$\lambda_y(\zeta) = (|y_1|^\zeta e^{i\theta_1}, \ldots, |y_n|^\zeta e^{i\theta_n});$$

note that $\lambda_y(1) = y$. Since $y \in \mathbf{C}^{*n} \setminus T$, λ_y is not constant. Set $\Delta = \{\zeta \in \mathbf{C} : |\zeta - 1| < 1/2\}$ and $\gamma_y = \lambda_y(\Delta) \subset \mathbf{C}^{*n}$. Then $\Phi(\lambda_y(\zeta)) = \operatorname{Re}\zeta \cdot \Phi(\lambda_y(1))$, so the restriction of Φ to γ_y is harmonic.

(b) follows from *(a)* since locally plurisubharmonic functions cannot charge pluripolar sets and $\Phi(y)$ is independent of $\arg y_k$, $1 \le k \le n$.

We will say that the unbounded locus of $u \in PSH(\mathbf{C}^n)$ is *separated at infinity* if there exists an exhaustion of $\mathbf{C}^n$ by bounded pseudoconvex domains Ω_k such that $\inf\{u(z) : z \in \partial\Omega_k\} > -\infty$ for each k. The collection of all functions $u \in \mathcal{L}$ whose unbounded loci are separated at infinity will be denoted by $\mathcal{L}_*$. By [3], Corollary 2.3, the Monge-Ampère current $(dd^c u)^n$ is well defined on $\mathbf{C}^n$ for any function $u \in \mathcal{L}_*$. Note also that $\Psi_{u,x} \in \mathcal{I}_0$ $\forall x \in \mathbf{C}^n$ for any $u \in \mathcal{L}_*$.

We are going to compare the total Monge-Ampère mass

$$M(u; \mathbf{C}^n) = \int_{\mathbf{C}^n} (dd^c u)^n$$

of $u \in \mathcal{L}_*$ with that of its indicator. The key result is the following comparison theorem (which is actually a variant of B.A. Taylor's theorem [21]).

Theorem 3 *Let $u, v \in \mathcal{L}_*$, $v \ge 0$ outside a bounded set, and*

$$\limsup_{z \to \infty} \frac{u(z)}{v(z) + \eta \log|z|} \le 1 \quad \forall \eta > 0.$$

Then $M(u; \mathbf{C}^n) \le M(v; \mathbf{C}^n)$.

Proof. By the definition of the class $\mathcal{L}_*$, there exist numbers $0 \le m_1 \le m_2 \le \ldots$ such that $u(z) > -m_k$ near $\partial\Omega_k$, $\{\Omega_k\}$ being the pseudoconvex exhaustion of $\mathbf{C}^n$. Let $w(z) = \sup\{u(z), -m_k\}$ for $\Omega_k \setminus \Omega_{k-1}$ (assuming $\Omega_0 = \emptyset$), then $w \in \mathcal{L} \cap L^\infty_{loc}(\mathbf{C}^n)$ and satisfies the same asymptotic relation at infinity as u does. Besides,

$$\int_{\Omega_k} (dd^c u)^n = \int_{\Omega_k} (dd^c w)^n \quad \forall k,$$

so

$$M(u; \mathbf{C}^n) = M(w; \mathbf{C}^n). \tag{17}$$

Denote $v_\eta(z) = v^+(z) + \eta \log^+|z|$, $\eta > 0$. Let $\epsilon > 0$ and $C > 0$, then $w(z) \le (1+\epsilon)v_\eta - 2C$ for all $z \in \mathbf{C}^n \setminus B_\alpha$ with B_α a ball of the radius $\alpha = \alpha(\eta, C, \epsilon)$. Therefore,

$$E(\eta, C, \epsilon) := \{z \in \mathbf{C}^n : (1+\epsilon)v_\eta - C < w(z)\} \subset\subset B_\alpha.$$

By the Comparison Theorem for bounded plurisubharmonic functions,

$$\int_{E(\eta,C,\epsilon)} (dd^c w)^n \le \int_{E(\eta,C,\epsilon)} (dd^c[(1+\epsilon)v_\eta - C])^n \le (1+\epsilon)\int_{\mathbf{C}^n} (dd^c v_\eta)^n. \tag{18}$$

For any compact $K \subset \mathbf{C}^n$ one can find $C > 0$ such that $K \subset E(\eta, C, \epsilon)$, so (18) gives us

$$M(w;\mathbf{C}^n) \le (1+\epsilon)M(v_\eta;\mathbf{C}^n).$$

Since v_η decreases to v^+ as $\eta \to 0$ and in view of the arbitrary choice of ϵ, we then get

$$M(w;\mathbf{C}^n) \le M(v^+;\mathbf{C}^n) = M(v;\mathbf{C}^n),$$

which by (17) completes the proof.

As an immediate consequence we have

Theorem 4 *For any* $u \in \mathcal{L}_*$,

$$M(u;\mathbf{C}^n) \le M(\Psi_{u,x};\mathbf{C}^n) \le M(\Psi_u;\mathbf{C}^n).$$

To get effective bounds for $M(u;\mathbf{C}^n)$, we estimate the Monge-Ampère masses of the indicators.

Proposition 9 *Let* $\Phi \in \mathcal{I},\ z^0 \in D^{-1}$. *Then*

$$\Phi(z) \le \Phi(z^0)\sup_k \frac{\log^+|z_k|}{\log|z_k^0|} \quad \forall z \in \mathbf{C}^n.$$

Proof. Denote $\psi = [\Phi(z^0)]^{-1}Exp^*\Phi,\ a = Log(z^0) \in \mathbf{R}_+^n$. It suffices to prove the relation $\psi(t) \le s_a^+(t)$ for all t with $|t_k| < a_k$, $1 \le k \le n$; here $s_a = Exp^*S_a$ with S_a defined in (7), so $s_a^+(t) = \sup_k t_k^+/a_k$.

We fix such a point t and denote $\alpha = s_a^+(t) < 1$, $\beta = (1-\alpha)^{-1} > 1$. Consider the segment $l_t = \{a + \lambda\,(t-a):\ 0 \le \lambda \le \beta\} \subset \mathbf{R}^n$; observe that

$$a_k + \beta\,(t_k - a_k) = \beta\, a_k\,(t_k a_k^{-1} - \alpha) \le 0,\ 1 \le k \le n. \tag{19}$$

Let $u(\lambda)$ be the restriction of ψ to l_t, $v(\lambda) = (\alpha-1)\lambda + 1$. The function u is convex on l_t, and v is linear. Besides, $u(0) = v(0) = 1$, $v(\beta) = 0$, and $u(\beta) \le 0$ in view of (19). Therefore, $u \le v$ on l_t. In particular, $\psi(t) = u(1) \le v(1) = s_a^+(t)$, and the proposition is proved.

Theorem 5 *Let $\Phi \in \mathcal{I}_0,\ z^0 \in D^{-1}$. Then*

$$M(\Phi; \mathbf{C}^n) \leq \frac{[\Phi(z^0)]^n}{\log|z_1^0|\ldots\log|z_n^0|}.$$

In particular, for any $u \in \mathcal{L}$,

$$M(\Psi_u; \mathbf{C}^n) \leq \frac{[\nu(u,a,\infty)]^n}{a_1\ldots a_n} \quad \forall a \in \mathbf{R}_+^n.$$

Proof. The first relation follows from Proposition 9 and Theorem 3, since (taking $a = Log(z^0) \in \mathbf{R}_+^n$)

$$M(S_a^+; \mathbf{C}^n) = M(S_a; \mathbf{C}^n) = [\log|z_1^0|\ldots\log|z_n^0|]^{-1}.$$

The second inequality results now from Proposition 3 *(b)*.

We can give a geometric interpretation for the total Monge-Ampère masses of indicators, which in many cases leads to their exact calculation.

Let $\Phi \in \mathcal{I},\ \varphi = Exp^*\Phi$. Denote

$$\Theta_\Phi^+ = \{a \in \mathbf{R}^n : \langle a,t\rangle \leq \varphi^+(t) \quad \forall t \in \mathbf{R}^n\}.$$

Proposition 10 *Θ_Φ^+ is a convex compact subset of $\overline{\mathbf{R}_+^n}$, $\Theta_\Phi^+ \subset \{a \in \mathbf{R}^n : 0 \leq a_k \leq \varphi^+(\mathbf{1}_k),\ 1 \leq k \leq n\}$.*

Proof. Convexity of Θ_Φ^+ is evident. Further, if $a \in \Theta_\Phi^+$, then $\langle a, \pm\mathbf{1}_k\rangle \leq \varphi^+(\pm\mathbf{1}_k)$, $\mathbf{1}_k$ being defined by (11), and the statement follows because $\varphi^+(-\mathbf{1}_k) = 0$.

By $Vol(P)$ we denote the Eucledean volume of $P \subset \mathbf{R}^n$.

Theorem 6 *For any $\Phi \in \mathcal{I}_0$,*

$$M(\Phi; \mathbf{C}^n) = n!\, Vol(\Theta_\Phi^+).$$

Proof. By Proposition 8,

$$M(\Phi; \mathbf{C}^n) = M(\Phi^+; \mathbf{C}^n) = \int_T (dd^c\Phi^+)^n.$$

It can be easily checked that the complex Monge-Ampère operator $(dd^cU)^n$ of an n-circled locally bounded plurisubharmonic function U is related to the real Monge-Ampère operator $\mathcal{MA}[u]$ of its convex image u by the equation

$$\int_G (dd^cU)^n = n! \int_{Log(G)} \mathcal{MA}[u]$$

for every n-circled Borelean set $G \subset \mathbf{C}^n$ (see e.g. [19]). Since $Log(T) = \{0\}$,

$$M(\Phi^+; \mathbf{C}^n) = n!\, \mathcal{MA}[\varphi^+](\{0\}). \tag{20}$$

As was established in [18], for any convex function v on a domain $\Omega \subset \mathbf{R}^n$,

$$\mathcal{MA}[v](F) = Vol\,(\omega(F, v)) \quad \forall F \subset \Omega, \tag{21}$$

where

$$\omega(F, v) = \bigcup_{t^0 \in F} \{a \in \mathbf{R}^n :\ v(t) \geq v(t^0) + \langle a, t - t^0\rangle\ \forall t \in \Omega\}$$

is the gradient image of the set F for the surface $\{y = v(x),\ x \in \Omega\}$. In our situation, it means that

$$\omega(\{0\}, \varphi^+) = \{a \in \mathbf{R}^n :\ \varphi^+(t) \geq \langle a, t\rangle\ \forall t \in \mathbf{R}^n\} = \Theta^+_\Phi, \tag{22}$$

so the statement follows from (20)–(22).

The set Θ^+_Φ for $\Phi = \Psi_{u,x},\ u \in \mathcal{L},\ x \in \mathbf{C}^n$, will be denoted by $\Theta^+_{u,x}$, and by Θ^+_u for $\Phi = \Psi_u$. Then Theorems 4 and 6 give us

Theorem 7 *For any $u \in \mathcal{L}_*$ and $x \in \mathbf{C}^n$,*

$$M(u; \mathbf{C}^n) \leq n!\, Vol(\Theta^+_{u,x}) \leq n!\, Vol(\Theta^+_u). \tag{23}$$

Remark. Let $u = \log|P|,\ P = (P_1, \ldots, P_N)$ being a polynomial mapping. By Proposition 6,

$$\psi_{u,x}(t) = I_{up}(P, x, t) = \sup_{1 \leq j \leq N} I_{up}(P_j, x, t),$$

the upper indices $I_{up}(P_j, x, t)$ defined by (13). In this case,

$$\Theta^+_{u,x} = \{a \in \mathbf{R}^n :\ \langle a, t\rangle \leq I^+_{up}(P, x, t)\ \forall t \in \mathbf{R}^n\},$$

so $\Theta^+_{u,0}$ coincides with the Newton polyhedron for P at infinity (see Introduction). If $N = n$ and $P^{-1}(0)$ is discrete, then $M(u; \mathbf{C}^n)$ is the number of zeros of P counted with the multiplicities. For this case, (23) gives the bound due to Kouchnirenko [7].

Theorem 7 produces also an upper bound for $M(u; \mathbf{C}^n)$ via the multitype $(\sigma_1(u), \ldots, \sigma_n(u))$ of the function u:

Theorem 8 *Let $u \in \mathcal{L}_*$, then*

$$M(u; \mathbf{C}^n) \leq n!\, \sigma_1(u) \ldots \sigma_n(u); \tag{24}$$

in particular,

$$\sum_x [\nu(u, x)]^n \leq n!\, \sigma_1(u) \ldots \sigma_n(u). \tag{25}$$

Proof. By Propositions 3 and 10, $\Theta^+_u \subset \{a \in \mathbf{R}^n :\ 0 \leq a_k \leq \sigma_k(u),\ 1 \leq k \leq n\}$, and (24) follows from Theorem 7. It implies (25) in view of the known inequality $(dd^c u)^n|_{\{x\}} \geq [\nu(u, x)]^n$.

Remark. It can be shown that inequality (24) implies Dyson's lemma for algebraic hypersurfaces with isolated singular points (see [22]).

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